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1998 J. Phys. A: Math. Gen. 31 3999

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J. Phys. A: Math. Gen. **31** (1998) 3999–4011. Printed in the UK PII: S0305-4470(98)88809-4

Effective potential for ϕ^6 theory in three-dimensional curved spacetime

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Received 28 October 1997

Abstract. We calculate the effective potential for a gauged ϕ^6 scalar field theory with minimal coupling to gravity, in $(2+1)$-dimensional curved spacetime using mean-field-theory approximation techniques. Gauge independence of the off-shell effective action is ensured by working within the framework of the Vilkovisky–DeWitt approach.

1. Introduction

It has long been of interest to study effective actions in quantum field theory as these tell us something of the structure of the vacuum which is probed by an external source [1, 2].

The formalism is useful, in particular, in the inflationary universe scenario, since the global minimum of the effective potential gives the exact ground-state energy of the system. It is also useful while studying particle creation in specific curved spacetimes, since the trace anomaly can be derived from the effective action [3]. The effective action concept has also found applications in Adler's induced gravity approach to quantum gravity [4] (see also [5] in this connection).

A feature of the conventional definition of the effective action is that it is gauge dependent off the mass shell. A few years ago Vilkovisky [6] and DeWitt [7, 8] gave a construction for the effective action which is gauge invariant, reparametrization invariant and gauge condition independent, even off the mass shell. It has been shown by Burgess and Kunstatter [9] that the Vilkovisky–DeWitt effective action is a member of a larger family of effective actions which are interpreted as the minimum energy of the field theory. They conclude that all the members of the family may be used to probe the vacuum field states with the same accuracy, the Vilkovisky–DeWitt effective action only producing simpler calculations.

Even if the effective action is gauge-independent off-shell, two other features persist—convexity, and the problem of having in hand a perturbative loop expansion which is ill defined for situations involving spontaneous breakdown of symmetry. The problem of convexity relates to the fact that whereas the exact effective potential is convex [10], which means that even for a theory with symmetry breaking at the classical level the exact effective potential can never have a double-well shape, its perturbative expansion is, however, neither

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0305-4470/98/173999+13$19.50 © 1998 IOP Publishing Ltd

convex nor real. This leads to an unsatisfactory situation as the exact and the perturbatively expanded effective potentials show different behaviour. The convexity problem has been treated by several authors, including O'Raifeartaigh and Parravicini [11] and Stevenson [12], the latter using the Gaussian effective potential approach, and more recently by Balakrishnan and Moss [13] who applied their formalism to the standard electroweak theory at high temperatures.

In $(2+1)$ dimensions, in flat spacetime, several interesting phenomena, such as the quantum Hall effect and high-temperature superconductivity, all involving planar gauge theoretic dynamics have been studied. The most general renormalizable relativistic scalar field theory in this spacetime is a ϕ^6 theory [14]. In curved $(2+1)$-dimensional spacetime, Einstein's theory of gravitation exhibits several unexpected features [15, 16]. As discussed in [15], in these dimensions the gravitational field has no dynamical degrees of freedom and one can obtain a quantum theory of the vacuum gravitational field only by coupling the metric to a source field having its own dynamical degrees of freedom.

In view of this, it is interesting to calculate the effective potential for a quantum field theory in these dimensions in order to obtain the quantum corrections to the classical field theory in the presence of gravity. The purpose of our paper is to study gauged ϕ^6 theory in a three-dimensional curved spacetime. Our aim is to include contributions from all the saddle points in the effective potential, and to do so in a gauge-independent manner. We consider only the case where the scalar field ϕ is coupled minimally to gravity and calculate the effective potential by making an expansion in powers of the curvature. We use zeta-function regularization to obtain a finite result.

Standard calculations of the effective potential making an expansion about only one saddle point suffer from the drawback that the naive loop expansion fails in the case of theories with broken symmetry. In a series of papers, Bender, Cooper, Guralnik and others [17] have developed a mean-field-theory approximation method for performing perturbative calculations leading to a well defined loop expansion for broken symmetric theories.

Their idea, essentially, is to introduce an auxiliary composite field and to rewrite the classical Lagrangian using the auxiliary field. They define their effective action as a double Legendre transform with two sources—one coupled to the expectation value of the scalar field and the other coupled to the expectation value of the auxiliary composite field. Their technique consists of performing the functional integration over all the scalar fields ϕ including all the saddle points, keeping ϕ^2 fixed. This gives a well defined perturbative loop expansion. The expansion is made in powers of a small parameter ϵ, and finally this value of ϵ is set to unity.

Although we employ this mean-field technique to calculate the one-loop effective potential, our procedure differs in certain small details. In order to maintain gauge condition independence throughout, even off the mass shell, we work within the framework of the covariant Vilkovisky–DeWitt approach.

2. Calculation of the effective potential

We begin with the classical action:

$$S = \int \mathrm{d}v_x \left\{ (D_\mu\phi)^\dagger (D^\mu\phi) + m^2\phi^\dagger\phi + \frac{\lambda}{6}(\phi^\dagger\phi)^2 + \frac{\eta}{90}(\phi^\dagger\phi)^3 \right\}. \qquad (1)$$

Here, $\mathrm{d}v_x = \sqrt{g}\,\mathrm{d}^3x$ is the invariant spacetime volume element and $D_\mu = \nabla_\mu + \mathrm{i}eA_\mu$ denotes the covariant derivative. The signature of the spacetime has been taken to be Riemannian.

We prefer to work with the polar coordinate parametrization for the complex scalar field ϕ

$$\phi(x) = \frac{1}{\sqrt{2}}\rho(x)\,\mathrm{e}^{\mathrm{i}\theta(x)} \tag{2}$$

so that the action can then be expressed in terms of the ρ and θ fields. The field redefinition [18]

$$A_\mu(x) = B_\mu(x) - \frac{1}{e}\partial_\mu\theta(x) \tag{3}$$

which some authors refer to as the 'unitary gauge choice', then eliminates the $\theta(x)$ field appearing in the resulting action and reduces the gauge theory to a non-gauge theory

$$S = \int \mathrm{d}v_x \,\left\{\frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \frac{1}{2}\rho(-\Box + m^2 + e^2 B_\mu B^\mu)\rho + \frac{\lambda}{4!}\rho^4 + \frac{\eta}{6!}\rho^6\right\} \tag{4}$$

where $\bar{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Following the mean-field approach of Bender *et al* [17], we introduce an auxiliary composite field $\chi(x)$ coupled to another external source $K(x)$. In terms of the field χ, the classical action can be rewritten as

$$\begin{aligned} S = \int \mathrm{d}v_x \bigg\{&\frac{1}{2}\rho\left[-\Box + m^2 + e^2 B_\mu B^\mu + \chi + \frac{3}{10}\frac{\eta}{\lambda^2}\chi^2\right]\rho \\ &+ \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{3}{2\lambda}\chi^2 - \frac{3}{5}\frac{\eta}{\lambda^3}\chi^3\bigg\} \end{aligned} \tag{5}$$

because the Euler–Lagrange equation for χ, $\delta\mathcal{L}/\delta\chi = 0$, yields $\chi = \lambda\rho^2/6$ for the positive root of χ, and substituting this value of χ in (5) gives us back (4). The field $\rho(x)$ now occurs in the Lagrangian (5) in powers no higher than quadratic.

The effective action $\Gamma_{\mathrm{eff}}[\bar{\phi}, \bar{\chi}]$ is now defined as the double Legendre transform

$$\Gamma_{\mathrm{eff}}[\bar{\phi}, \bar{\chi}] = -\hbar \ln Z[J, K] - \int \mathrm{d}v_x\, J(x)\bar{\phi}(x) - \int \mathrm{d}v_x\, K(x)\bar{\chi}(x) \tag{6}$$

where the partition function $Z[J, K]$ is defined as

$$Z[J, K] = \int \mathrm{d}\mu[\phi]\,\mathrm{d}\mu[\chi]\exp\left\{-\frac{1}{\hbar}\left[S[\phi, \chi] + \int \mathrm{d}v_x (J(x)\phi(x) + K(x)\chi(x))\right]\right\}. \tag{7}$$

The expectation values $\bar{\phi}$ and $\bar{\chi}$ are given by

$$-\hbar\frac{\delta \ln Z[J, K]}{\delta J(x)} = \bar{\phi}(x) \qquad -\hbar\frac{\delta \ln Z[J, K]}{\delta K(x)} = \bar{\chi}(x) \tag{8}$$

and the external sources $J(x)$ and $K(x)$ are defined by the effective field equations:

$$\frac{\delta\Gamma_{\mathrm{eff}}}{\delta\bar{\phi}(x)} = -J(x) \qquad \frac{\delta\Gamma_{\mathrm{eff}}}{\delta\bar{\chi}(x)} = -K(x). \tag{9}$$

In order to show the equivalence up to zeroth order in $\epsilon\hbar$ of the theories defined by (5) and (4), we now rewrite the partition function (7) as

$$Z = \int \mathrm{d}\mu[\phi]\exp\left\{-\frac{1}{\epsilon\hbar}\left(S_a[\phi] + \int \mathrm{d}v_x\, J\phi\right)\right\}\int \mathrm{d}\chi\,\exp\left\{-\frac{1}{\epsilon\hbar}Q(\chi, \rho)\right\} \tag{10}$$

where

$$\begin{aligned} S_a[\phi] &= \int \mathrm{d}v_x\,\left\{\frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \frac{1}{2}\rho(-\Box + m^2 + e^2 B_\mu B^\mu)\rho\right\} \\ Q[\rho, \chi] &= \int \mathrm{d}v_x\,\left\{-\frac{3}{2\lambda}\chi^2 - \frac{3\eta}{5\lambda^3}\chi^3 + \frac{1}{2}\rho^2\chi + \frac{3\eta}{20\lambda^2}\chi^2\rho^2 + K\chi\right\}. \end{aligned} \tag{11}$$

We can Taylor expand $Q(\rho, \chi)$ about values $\chi_\pm$ of χ

$$Q[\rho,\chi] = Q[\rho,\chi_\pm] + \frac{\delta Q}{\delta\chi}\bigg|_{\chi_\pm}(\chi-\chi_\pm) + \frac{1}{2}\frac{\delta^2 Q}{\delta\chi^2}\bigg|_{\chi_\pm}(\chi-\chi_\pm)^2 + \frac{1}{6}\frac{\delta^3 Q}{\delta\chi^3}\bigg|_{\chi_\pm}(\chi-\chi_\pm)^3 + \cdots \tag{12}$$

where $\chi_\pm$ satisfies the stationarity condition

$$\frac{\delta Q}{\delta\chi} = 0 \tag{13}$$

which has the roots

$$\chi_\pm = \frac{5\lambda^2}{6\eta}\left(\frac{\eta\rho^2}{10\lambda} - 1\right) \pm \frac{1}{2}\left[\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)^2 + \frac{20\lambda^3 K}{9\eta}\right]^{1/2}. \tag{14}$$

Choosing the positive root χ_+ gives

$$\begin{aligned} Q[\rho,\chi] = {} & \frac{\eta\rho^6}{6!} + \frac{\lambda^4}{4!} + \frac{5\lambda^3}{18\eta}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)^{-1} K^2 + \frac{\lambda\rho^2}{6}K \\ & - \frac{9\eta}{5\lambda^3}\left[\frac{1}{4}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)^2 + \frac{5\lambda^3 K}{9\eta}\right]^{1/2}(\chi-\chi_+)^2 - \frac{3\eta}{5\lambda^3}(\chi-\chi_+)^3 + \cdots . \end{aligned} \tag{15}$$

To get this, we have assumed that

$$\frac{20\lambda^3 K}{9\eta}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)^{-2} \ll 1. \tag{16}$$

If we neglect terms which are quadratic and higher order in K in comparison with those linear in K by making use of the assumption (16) made above, we obtain after performing the shift $\chi - \chi_+ \to \chi$ in the field variable χ

$$\begin{aligned} Z[J,K] = {} & \int \mathrm{d}\mu[\rho] \exp\left\{ -\frac{1}{\epsilon\bar{h}}\left(S_4 + \int \mathrm{d}v_x \left[J\rho + \frac{\lambda\rho^2}{6}K\right]\right)\right\} \\ & \times \int \mathrm{d}\mu[\chi] \exp\left\{ -\frac{1}{\epsilon\bar{h}} \int \mathrm{d}v_x \left\{ -\frac{9\eta}{5\lambda^3}\left[\frac{1}{4}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)^2 + \frac{5\lambda^3 K}{9\eta}\right]^{1/2}\chi^2 \right.\right. \\ & \left.\left. - \frac{3\eta}{5\lambda^3}\chi^3 + \cdots \right\}\right\} \end{aligned} \tag{17}$$

where S_4 is the classical action S defined in (4).

Setting the source K to zero leads to the result

$$\begin{aligned} Z[J,K] = {} & \int \mathrm{d}\mu[\rho] \exp\left\{ -\frac{1}{\epsilon\bar{h}}\left(S_4 + \int \mathrm{d}v_x\, J\rho\right)\right\} \times \int \mathrm{d}\mu[\chi] \\ & \times \exp\left\{ -\frac{1}{\epsilon\bar{h}} \int \mathrm{d}v_x \left[-\frac{9\eta}{10\lambda^3}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right)\chi^2 - \frac{3\eta}{5\lambda^3}\chi^3 + \cdots \right]\right\}. \end{aligned} \tag{18}$$

Thus to lowest (zeroth) order in $\epsilon\bar{h}$, the original ϕ^6 theory (with the Lagrangian (4)) is obtained and the Lagrangian in (5) is equivalent to (4). The χ integral contributes to the next order in the $\epsilon\bar{h}$ expansion.

The effective action defined in (6) bears a slight resemblance to that defined by Cornwall *et al* [19] for composite fields, but in fact differs from it, and so do the field equations (9) following from it. Equation (6) can be rewritten as

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{\chi}] = -\hbar \ln \int \mathrm{d}\mu[\phi]\, \mathrm{d}\mu[\chi] \times \exp\left\{ -\frac{1}{\hbar}\left[S[\phi, \chi] + \int \mathrm{d}v_x\, J(\phi - \bar{\phi}) + \int \mathrm{d}v_x\, K(\chi - \bar{\chi}) \right]\right\}. \tag{19}$$

The procedure now consists of performing the integration over all the scalar field configurations ϕ, including all saddle points, keeping χ fixed. The resulting quantity $F[\chi]$ is replaced by $(1/\epsilon)F[\chi]$, where ϵ is a small parameter ($\epsilon > 0$), and Laplace's method is used to evaluate the integral by performing an expansion of the integrand as a series in powers of ϵ. At the end of the calculations, ϵ is set to 1. The effective action can thus be calculated from

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{\chi}] = -\epsilon\hbar \ln \int \mathrm{d}\chi\, \mathrm{e}^{-(1/\epsilon)F[\chi]} \tag{20}$$

to different orders in the $\hbar$ loop expansion:

$$\Gamma_{\text{eff}} = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}. \tag{21}$$

We find it suitable to adopt the Vilkovisky–DeWitt procedure, since we have used polar coordinate parametrization for the fields ϕ. It can be shown that different parametrizations for the fields lead to different results for the effective action if sufficient care is not taken. A lucid account of this has been given by Toms [20].

We will now briefly go over the essence of the Vilkovisky–DeWitt (V–D) procedure before using it in our calculations. We use DeWitt's condensed notation [21] here. To start with, the V–D method considers all the fields ϕ^i as local coordinates of points on the field space $\mathcal{F}$ which is an infinite-dimensional manifold. The classical action $S[\phi]$ is invariant under the infinitesimal gauge transformations

$$\delta\phi^i = K^i{}_\alpha[\phi]\delta\epsilon^\alpha \tag{22}$$

where the infinitesimal parameters $\delta\epsilon^\alpha$ characterize the transformation and $K^i{}_\alpha$ are the gauge transformation generators. It can be shown that the problem of non-invariance of Γ_{eff} under reparametrizations in conventional quantum field theory arises because the source J_i is coupled linearly to the field ϕ^i. In this connection, see Kunstatter [22]. In the V–D formalism, this problem is remedied by replacing $J_i(\phi^i - \bar{\phi}^i)$ in the argument of the exponential in (19) by the geometrical entity $J_i\sigma^i[\bar{\phi}, \phi]$ where $\sigma^i[\bar{\phi}, \phi]$ is the tangent vector at $\bar{\phi}$ to the geodesic connecting $\bar{\phi}$ to ϕ. Furthermore, all the ordinary derivatives are replaced by covariant derivatives. We work only to one-loop order, and to this order Vilkovisky's definition of the effective action coincides with DeWitt's [20]. From (19) it is seen that all integrations except that over χ are contained in

$$I[\chi] = \int \mathrm{d}\mu\,[\phi] \exp\left\{ -\frac{1}{\hbar}\left[S_1[\phi, \chi] - \frac{\delta\Gamma}{\delta\bar{\phi}^i}\sigma^i[\bar{\phi}, \phi]\right]\right\}. \tag{23}$$

Here again we have used DeWitt's condensed notation for convenience and denoted all fields except the χ field by the generic symbol ϕ^i ($\bar{\phi}^i$ denote their expectation values).

In equation (23), we have rewritten the classical action in (5) for the sake of convenience as

$$S = S_1[\phi, \chi, B_\mu] + S_2[\chi] \tag{24}$$

where

$$S_1[\phi, \chi, B_\mu] = \int \mathrm{d}v_x \left\{ \frac{1}{2}\rho \left[-\Box + m^2 + e^2 B_\mu B^\mu + \chi + \frac{3}{10}\frac{\eta}{\lambda^2}\chi^2 \right] \rho + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} \right\}$$
$$S_2[\chi] = \int \mathrm{d}v_x \left\{ -\frac{3}{2\lambda}\chi^2 - \frac{3}{5}\frac{\eta}{\lambda^3}\chi^3 \right\}. \tag{25}$$

Performing a covariant Taylor expansion of S_1 about $\bar{\phi}^i$ and using (21) yields

$$I[\chi] = \exp\left\{ -\frac{1}{\epsilon\hbar}\left[S_1[\bar{\phi}^i, \chi] + \frac{\epsilon\hbar}{2} \ln\det S_1{}^{;i}{}_j \right] \right\} \tag{26}$$

where $S_1[\bar{\phi}, \chi]$ is the $n = 0$ term in the covariant Taylor series expansion of $S_1[\phi^i, \chi]$. The covariant derivative $S^{;i}{}_j[\phi]$ of $S[\phi]$ is defined as

$$S^{;i}{}_j[\phi] = g^{ik}(S_{,kj} - \bar{\Gamma}^m_{kj} S_{,m}) \tag{27}$$

where $g_{ij}[\phi]$ is the metric on the field space, S_m stands for the functional derivative $\delta S/\delta\phi^m$, and $\bar{\Gamma}^m_{kj}$ denotes the Christoffel connection for the orbit space $\mathcal{F}/\mathcal{G}$:

$$\bar{\Gamma}^k_{ij} = \Gamma^k{}_{ij} - \tfrac{1}{2} g^{km}[\gamma^{\alpha\beta} K_{\alpha j} K_{\beta m,i} + \gamma^{\alpha\beta} K_{\alpha i} K_{\beta m,j} - (\gamma^{\alpha\beta} K_{\alpha i} K_{\beta j})_{,m}]. \tag{28}$$

$\gamma_{\alpha\beta}$ is the metric on the group space $\mathcal{G}$ and is defined as

$$\gamma_{\alpha\beta} = K^i{}_\alpha[\phi] g_{ij}[\phi] K^j{}_\beta[\phi]. \tag{29}$$

$\Gamma^k{}_{ij}$ stands for the connection for the field space metric g_{ij}. It may be noted that in order to show that the $n = 1$ term in the series expansion of S_1 cancelled with the source term in (23), it is neither necessary to assume that $\bar{\phi}$ is a solution to the classical equations of motion, nor even that it is close to a classical solution [23].

Substituting $I[\chi]$ back into (19), we obtain using (25)

$$F[\chi] = \frac{1}{\hbar}\left[K(\chi - \bar{\chi}) + \frac{1}{2}\bar{\rho}\left(-\Box + m^2 + e^2 \bar{B}_\mu \bar{B}^\mu + \chi + \frac{3}{10}\frac{\eta}{\lambda^2}\chi^2 \right)\bar{\rho} + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} \right.$$
$$\left. -\frac{3}{2\lambda}\chi^2 - \frac{3}{5}\frac{\eta}{\lambda^3}\chi^3 + \frac{\epsilon\hbar}{2}(\ln\det S_{1;ij}(\chi))_{\bar{B}\bar{\rho}} \right]. \tag{30}$$

Making the assumption that we can perform a Taylor series expansion of $F[\chi]$ about χ_0 where χ_0 corresponds to that value of χ for which $F[\chi]$ is stationary,

$$F'[\chi_0] = 0 \tag{31}$$

where the prime denotes a functional derivative with respect to χ, we get

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{\chi}] = -\epsilon\hbar \ln \int \mathrm{d}\chi \exp\left\{ -\frac{1}{\epsilon}\left[F[\chi_0] + \frac{1}{2}F''(\chi - \chi_0)^2 + \frac{1}{6}F'''(\chi - \chi_0)^3 + \cdots \right] \right\}. \tag{32}$$

This simplifies to

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{\chi}] = \hbar F[\chi_0] + \frac{\epsilon\hbar}{2} \ln\det\left(\frac{F''[\chi]}{2} \right)_{\chi=\chi_0} + \cdots \tag{33}$$

where $F[\chi_0]$ stands for $F[\chi_0, \bar{\chi}]$ and in the second term above we have retained only those terms of F which are quadratic in χ. The stationarity condition (31) helps us to determine the value of the source K to be

$$K = \frac{9\eta}{5\lambda^3}\chi_0^2 + \frac{3}{\lambda}\chi_0 - \frac{1}{2}\left(1 + \frac{3\eta}{5\lambda^2}\chi_0 \right)\bar{\rho}^2 - \frac{\epsilon\hbar}{2}\left\{ \frac{\delta}{\delta\chi}(\ln\det S_1{}^{;i}{}_j(\chi, \bar{B}, \bar{\rho})) \right\}_{\chi=\chi_0}. \tag{34}$$

Upon doing the χ integration we obtain finally

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{\chi}] = \frac{1}{2}\bar{\rho}\left(-\Box + m^2 + \chi_0 + e^2 B_\mu B^\mu + \frac{3\eta}{10\lambda^2}\chi_0^2\right)\bar{\rho} + \frac{\epsilon\hbar}{2}(\ln\det S_{1;ij}(\chi_0, \bar{B}, \bar{\rho}))$$
$$-\frac{3}{2\lambda}\chi_0^2 - \frac{3\eta}{5\lambda^3}\chi_0^3 + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + K(\chi_0 - \bar{\chi})$$
$$+\frac{\epsilon\hbar}{2}\ln\det\left(\frac{3\eta}{10\lambda^2}\bar{\rho}^2 - \frac{3}{\lambda} - \frac{18\eta}{5\lambda^3}\chi_0\right) + \cdots. \quad (35)$$

For the action (5), the components of the generators of the gauge transformation may be obtained from (22):

$$K^{\rho(x)}{}_{x'} = 0 \qquad K^{\theta(x)}{}_{x'} = e\delta(x, x')$$
$$K^{A_\mu(x)}{}_{x'} = -\nabla_\mu\delta(x, x') \qquad K^{\chi(x)}{}_{x'} = 0. \quad (36)$$

Note that χ are gauge-invariant fields. It is seen that the components of the field-space metric are given by

$$g_{\rho(x)\rho(x')} = \delta(x, x') \qquad g_{\theta(x)\theta(x')} = \rho^2(x)\delta(x, x')$$
$$g_{A_\mu(x)A_\nu(x')} = g^{\mu\nu}(x)\delta(x, x'). \quad (37)$$

The uncondensed notation used here is that followed by Kunstatter [24].

Since the χ field is gauge invariant, it does not contribute at all either to the field-space or to the orbit-space connections. It can be explicitly shown that the non-zero components of the Christoffel connection in the orbit space $\mathcal{F}/\mathcal{G}$ relevant to us are

$$\bar{\Gamma}^{\rho(x)}_{B_\nu(x')B_\lambda(x'')} = -e^2\rho(x)\nabla'_\nu\gamma^{x'x}\nabla''_\lambda\gamma^{x''x}$$
$$\bar{\Gamma}^{B_\mu(x)}_{\rho(x')B_\lambda(x'')} = \nabla_\mu\left(\frac{1}{\rho(x)}\delta(x, x')\nabla''_\lambda\gamma^{x''x}\right). \quad (38)$$

Here, $\gamma^{xx'}$ satisfies

$$[-\Box_x + e^2\rho^2(x)]\gamma^{xx'} = \delta(x, x') \quad (39)$$

and is the inverse of the group space metric $\gamma_{xx'}$ defined in (29). The covariant derivatives in (38) operate only on the first argument in γ^{xy}. The procedure which now follows to evaluate $\ln\det S_{1;i}{}^j$ is similar in spirit to that in [25] where the V–D procedure was discussed for ϕ^4 theory. As in the usual effective potential calculations, we set the fields $\bar{\rho} = \rho =$ constant and the background field $\bar{B}_\mu = 0$. After some amount of work, it can be shown that

$$\ln\det S_{1;i}{}^j = \ln\det\left(-\Box + m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right) + \ln\det[(-\Box + e^2\rho^2)g_{\mu\nu} + R_{\nu\mu}].$$
$$-\ln\det(-\Box + e^2\rho^2) + \ln\det[\delta^\mu{}_\nu\delta(x, x') + P^\mu{}_\nu(x, x')]. \quad (40)$$

We have arrived at this result after making use of the fact that the Faddeev–Popov factor cancels with the factor arising from the functional integration over θ, in the measure [26]. The quantity $P^\mu{}_\nu$ is given by

$$P^\mu{}_\nu(x, x') = ve^2\rho^2\int \mathrm{d}v_y\,\mathrm{d}v_z\, G^\mu{}_\lambda(x, z)\nabla^\lambda{}_z\gamma^{zy}\nabla'_\nu\gamma^{yx'}\left(m^2 + \chi(y) + \frac{3\eta}{10\lambda^2}\chi^2(y)\right) \quad (41)$$

where the Green function $G^\mu{}_\nu(x, y)$ satisfies

$$((-\Box + e^2\rho^2)g_{\mu\lambda} + \nabla_\lambda\nabla_\mu)G^{\lambda\nu}(x, y) = \delta_\mu{}^\nu\delta(x, y). \quad (42)$$

Further simplification leads to the result

$$
\begin{aligned}
\ln\det S_{1;i}{}^{j} = {} & \ln\det\left[\mu^{-2}\left(-\Box + m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)\right] \\
& + \ln\det[\mu^{-2}((-\Box + e^2\rho^2)g_{\mu\nu} + R_{\nu\mu})] \\
& - 3\ln\det[\mu^{-2}(-\Box + e^2\rho^2)] + \ln\det[\mu^{-2}(-\Box + e^2\rho^2 + b_1)] \\
& + \ln\det[\mu^{-2}(-\Box + e^2\rho^2 + b_2)] + \mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} f^{(n)}
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
b_1 = {} & \frac{v}{2}\left(m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\left\{\left(m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\right. \\
& \left. + \left(4e^2\rho^2 + m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\right\} \\
b_2 = {} & \frac{v}{2}\left(m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\left\{\left(m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\right. \\
& \left. - \left(4e^2\rho^2 + m^2 + \chi + \frac{3\eta}{10\lambda^2}\chi^2\right)^{1/2}\right\}
\end{aligned}
\tag{44}
$$

and $f^{(n)}$ are terms all of which contain spacetime (covariant) derivatives of χ. The $f^{(n)}$ arise from the last term on the right-hand side of (40). The quantity μ has dimensions of mass and has been inserted to make the arguments of the logarithms dimensionless. The factor v takes the value 1 in the V–D approach and zero in conventional quantum field theory.

We now evaluate all the terms of the form $\ln\det(-\Box + m^2 + Q)$ in the expression above using the generalized zeta-function technique [27]. One can show that

$$
\ln\det[(-\Box + m^2 + Q)\mu^{-2}] = -\zeta'(0) + \mu^{-2}\zeta(0) \tag{45}
$$

where the generalized zeta function $\zeta(s)$ is defined as

$$
\zeta(s) = \sum_{n=1}^{\infty}\lambda_n^{-s} \tag{46}
$$

which may also be written as

$$
\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{d}\tau\, \tau^{s-1}\, \mathrm{e}^{-m^2\tau} K(\tau) \tag{47}
$$

where λ_n are the eigenvalues of the operator $(-\Box + Q)$. The heat kernel $K(\tau)$ has a known asymptotic expansion of the form

$$
K(\tau) \sim (4\pi\tau)^{-d/2}\sum_{k=0}^{\infty}\tau^k \int \mathrm{d}v_x\, \mathrm{tr}E_k(x) \tag{48}
$$

in the limit $\tau \to 0$, where $E_k(x)$ are known coefficients, known in the literature as Gilkey coefficients [21, 28]. Using (48) in (47), we can evaluate all but the last term in (43) by making an expansion in powers of the curvature. Our calculations have been carried out up to third order in the curvature.

It is well known that in three-dimensional spacetime the Riemann curvature tensor can be expressed in terms of the Ricci tensor, as they both have the same number (six) of algebraically independent components:

$$
R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta} + \tfrac{1}{2}R(g_{\beta\gamma}g_{\alpha\delta} - g_{\beta\delta}g_{\alpha\gamma}). \tag{49}
$$

Using these results we obtain after a little work

$$\begin{aligned}\ln\det S_1{}^{;i}{}_j(x,\rho) &= \mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}f^{(n)} - \frac{1}{8\pi}\int \mathrm{d}v_x\,\mathrm{tr}\Bigg\{\frac{4}{3}\left(m^2+\chi+\frac{3\eta}{10\lambda^2}\chi^2\right)^{3/2}\\
&\quad+\frac{4}{3}[(e^2\rho^2+b_1)^{3/2}+(e^2\rho^2+b_2)^{3/2}]+R\Bigg(2e\rho-\frac{1}{3}\left(m^2+\chi+\frac{3\eta}{10\lambda^2}\right)^{1/2}\\
&\quad-\frac{1}{3}[(e^2\rho^2+b_1)^{1/2}+(e^2\rho^2+b_2)^{1/2}]\Bigg)+\left(\frac{1}{40}R^2+\frac{1}{60}R_{\mu\nu}R^{\mu\nu}+\frac{1}{30}\Box R\right)\\
&\quad\times\left[\frac{1}{(m^2+\chi+(3\eta/10\lambda^2)\chi^2)^{1/2}}+\frac{1}{e^2\rho^2}[(e^2\rho^2+b_1)^{1/2}+(e^2\rho^2+b_2)^{1/2}]\right]\\
&\quad+\Bigg[\frac{1}{2(m^2+\chi+(3\eta/10\lambda^2)\chi^2)^{3/2}}+\frac{1}{(e^2\rho^2)^3}[(e^2\rho^2+b_1)^{3/2}\\
&\quad+(e^2\rho^2+b_2)^{3/2}]\Bigg]E_{3c}\\
&\quad+\frac{1}{e\rho}\left(-\frac{R^2}{3}+\frac{1}{6}R_{\mu\nu}R^{\mu\nu}-\frac{1}{6}\Box R\right)+\frac{1}{2(e\rho)^3}E'_3+\cdots\Bigg\}\end{aligned}\tag{50}$$

where

$$\begin{aligned}E_{3c} &= \frac{1}{7!}\Big[18\Box^2R-10R_{;\mu}R^{;\mu}+34R_{\mu\nu;\rho}R^{\mu\nu;\rho}-4R_{\mu\nu;\rho}R^{\mu\rho;\nu}+24R_{\mu\nu}R^{\mu\rho;\nu}{}_{\rho}\\
&\quad+16R\Box R-32R_{\mu\nu}\Box R^{\mu\nu}-9R^3+118RR_{\mu\nu}R^{\mu\nu}-\frac{6928}{90}R_\mu{}^\alpha R_\alpha{}^\beta R_\beta{}^\mu\Big]\\
E'_3 &= \frac{1}{30}R_{;\alpha}R^{;\alpha}-\frac{1}{60}R_{\mu\lambda;\alpha}R^{\lambda\mu;\alpha}+\frac{1}{90}R_{\gamma\alpha;\beta}R^{\beta\alpha;\gamma}-\frac{1}{180}R_\gamma{}^\beta{}_{;\beta}R^{;\gamma}-\frac{1}{180}R_{;\beta}R^\beta{}_\gamma{}^{;\gamma}\\
&\quad-\frac{4}{45}R^3+\frac{1}{10}R^\mu{}_\alpha\Box R^\alpha{}_\mu-\frac{1}{90}R^{\alpha\beta}R_{;\alpha\beta}-\frac{1}{60}\Box^2R-\frac{17}{180}R\Box R\\
&\quad+\frac{1}{9}R^\mu{}_\alpha R^\alpha{}_\beta R^\beta{}_\mu+\frac{43}{180}RR_{\mu\nu}R^{\mu\nu}.\end{aligned}\tag{51}$$

In order to express the effective action for the scalar field as a function of $\bar{\phi}$ only, it is necessary to eliminate the auxiliary field $\bar{\chi}$. A possible way of achieving this is by setting the source K in the effective field equations (9) to zero. From (34) we then obtain the equation

$$\frac{9\eta}{5\lambda^3}\chi_0^2+\frac{3}{\lambda}\chi_0-\frac{1}{2}\left(1+\frac{3\eta}{5\lambda^2}\chi_0\right)\rho^2-\frac{\epsilon\bar{h}}{2}\left(\frac{\delta}{\delta\chi}(\ln\det S_1{}^{;i}{}_j(\chi,\rho))\right)_{\chi=\chi_0}=0.\tag{52}$$

Although one can obtain a solution for χ from here as a power series in ϵ, we solve for χ only to lowest order in ϵ, for the sake of simplicity, and consider the root

$$\chi_0=\frac{\lambda}{6}\rho^2.\tag{53}$$

Substituting for χ_0 from here into Γ_{eff} in (35), we obtain with the help of (34), to lowest order in ϵ,

$$\Gamma_{\mathrm{eff}}{}^{(0)}[\bar{\phi}]=\frac{1}{2}m^2\rho^2+\frac{\lambda}{24}\rho^4+\frac{\eta}{720}\rho^6.\tag{54}$$

Thus the tree-level and lowest-order result in ϵ coincides with the classical action we started with.

It is interesting to note that the equivalence of the Lagrangians (5) and (4) up to zeroth order in $\epsilon\hbar$ can be shown in the following alternative manner, starting from our result for the effective action (equation (35)).

$\bar{\chi}$ occurring in the K term in equation (35) can be found from (8), and by making a Taylor expansion in (7) about the value χ_0 of χ, where we have defined χ_0 in (31). This gives

$$\bar{\chi} = -\epsilon\hbar\frac{\delta \ln Z}{\delta K} = \chi_0 + \mathrm{O}(\epsilon\hbar) \tag{55}$$

where $\mathrm{O}(\epsilon\hbar)$ denotes terms which are of order one and higher in $\epsilon\hbar$. Substituting this back into (35), we obtain

$$\begin{aligned}\Gamma_{\mathrm{eff}}[\bar{\phi}, \bar{\chi}] = \frac{1}{2}\bar{\rho}\left(-\Box + m^2 + \chi_0 + e^2 B_\mu B^\mu + \frac{3\eta}{10\lambda^2}{\chi_0}^2\right)\bar{\rho} + \frac{\epsilon\hbar}{2}(\ln\det S_{1;ij}(\chi_0, \bar{B}, \bar{\rho}))\\ -\frac{3}{2\lambda}\chi_0^2 - \frac{3\eta}{5\lambda^3}\chi_0^3 + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + K\cdot\mathrm{O}(\epsilon\hbar)\\ +\frac{\epsilon\hbar}{2}\ln\det\left(\frac{3\eta}{10\lambda^2}\bar{\rho}^2 - \frac{3}{\lambda} - \frac{18\eta}{5\lambda^3}\chi_0\right) + \cdots.\end{aligned} \tag{56}$$

Collecting together from here all terms which are zeroth order in $\epsilon\hbar$, one can try to generate the Green functions for the ϕ-theory (with external ϕ lines) from the integral

$$\begin{aligned}Z[J,K] = \exp\left\{-\frac{1}{\epsilon\hbar}W[J]\right\} = \int \mathrm{d}\mu[\rho]\,\mathrm{d}\mu[\chi]\\ \times\exp\left\{-\frac{1}{\epsilon\hbar}\int \mathrm{d}v_x\left[\frac{1}{2}\rho(-\Box + m^2 + B_\mu B^\mu)\rho + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + J\rho\right]\right\}\\ \times\exp\left\{-\frac{1}{\epsilon\hbar}U[\rho,\chi]\right\}\end{aligned} \tag{57}$$

where

$$U[\rho,\chi] = \int \mathrm{d}v_x\left\{-\frac{3}{2\lambda}\chi^2 - \frac{3\eta}{5\lambda^3}\chi^3 + \frac{\rho^2}{2}\chi + \frac{3\eta}{20\lambda^2}\chi^2\rho^2\right\}. \tag{58}$$

One can perform a Taylor expansion of $U[\rho,\chi]$ about the values $\chi_\pm$ of χ as before

$$U[\rho,\chi] = U[\rho,\chi_\pm] + \left.\frac{\delta U}{\delta\chi}\right|_{\chi_\pm}(\chi-\chi_\pm) + \frac{1}{2}\left.\frac{\delta^2 U}{\delta\chi^2}\right|_{\chi_\pm}(\chi-\chi_\pm)^2 + \frac{1}{6}\left.\frac{\delta^3 U}{\delta\chi^3}\right|_{\chi_\pm}(\chi-\chi_\pm)^3 + \cdots \tag{59}$$

where U satisfies the stationarity condition

$$\frac{\delta U}{\delta\chi} = 0 \tag{60}$$

for the values $\chi_\pm$ of χ. The roots $\chi_\pm$ are given by

$$\chi_\pm = \left(\frac{\lambda\rho^2}{12} - \frac{5\lambda^2}{6\eta}\right) \pm \frac{1}{2}\left(\frac{\lambda\rho^2}{6} + \frac{5\lambda^2}{3\eta}\right). \tag{61}$$

We choose the positive root $\chi_+ = \lambda\rho^2/6$ which after substitution into (59) gives

$$U[\chi,\rho] = \frac{\lambda\rho^4}{4!} + \frac{\eta\rho^6}{6!} - \frac{3\eta}{10\lambda^3}\left(\frac{\lambda\rho^2}{2} + \frac{5\lambda^2}{\eta}\right)(\chi-\chi_+)^2 - \frac{3\eta}{5\lambda^3}(\chi-\chi_+)^3 + \cdots. \tag{62}$$

As before, performing the shift in the χ variable $\chi - \chi_+ \to \chi$, and substituting $U[\chi,\rho]$ back into (57) results in the expression (18).

When the scalar field ϕ has a large (N) number of components, the replacements $\lambda \to \lambda/N$ and $\eta \to \eta/N^2$ in (4) would yield the classical Lagrangian for the theory. In this case, it is easy to check by repeating the above line of reasoning that we would obtain

$$\begin{aligned} Z[J] = \int \mathrm{d}\mu[\phi] \exp\Bigg\{ -\frac{1}{\epsilon\hbar} \int \mathrm{d}v_x \\ &\times \left[\frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \frac{1}{2}\phi(-\Box + m^2 + e^2 B_\mu B^\mu)\phi + \frac{\lambda}{4!}\frac{\phi^4}{N} + \frac{\eta\phi^6}{6!N^2} + J\phi\right]\Bigg\} \\ &\times \int \mathrm{d}\mu\,[\chi] \exp\Bigg\{ -\frac{1}{\epsilon\hbar}\int \mathrm{d}v_x \, \frac{1}{(1/N)} \\ &\times \left[-\frac{9\eta}{10\lambda^3}\left(\frac{\lambda\rho^2}{6N} + \frac{5\lambda^2}{3\eta}\right)\chi^2 - \frac{3\eta}{5\lambda^3}\chi^3 + \cdots \right]\Bigg\}. \end{aligned} \tag{63}$$

Thus it is seen that the ϕ integral alone generates the Green functions for ϕ^6 theory, while the extra terms generated by the χ integral lead to an expansion in powers of $1/N$.

It was explained in [17] for ϕ^6 theory that the limit $N \to \infty$ corresponds to the limit $\epsilon \to 0$. We have thus been able to show that the original ϕ^6 theory is regained at least to a first approximation, i.e. if the contributions from the χ integral which are next to leading order in $1/N$ are ignored.

We shall now consider symmetry breaking. Townsend [14] observed that symmetry breaking for ϕ^6 theory occurs for three cases: (1) $\lambda \geqslant 0, m^2 < 0$, (2) $\lambda < 0, m^2 < 0$, and (3) $\lambda < 0, m^2 \geqslant 0$, provided the parameters λ, m^2, and η were restricted to certain values. We consider first the case for which $m^2 > 0$ and $\lambda < 0$. Substituting the value of χ_0 from (52) into (35) and making the replacement $\lambda \to -\lambda'$, we find that apart from constants

$$\Gamma_{\mathrm{eff}}[\bar{\phi}] = \frac{1}{2}m^2\rho^2 - \frac{\lambda'}{4!}\rho^4 + \frac{\eta}{6!}\rho^6 + \frac{\epsilon\hbar}{2}\left[\mathrm{Tr}\ln\left(1 - \frac{\eta}{10\lambda'}\rho^2\right) + \ln\det S_1{}^{;i}{}_j[\rho]\right] \tag{64}$$

where

$$\begin{aligned} \ln\det S_1{}^{;i}{}_j[\rho] = -\frac{1}{8\pi}\int \mathrm{d}v_x \,\mathrm{Tr}\Bigg\{ \frac{4}{3}\left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{3/2} \\ &+ \frac{4}{3}[(e^2\rho^2 + a_1)^{3/2} + (e^2\rho^2 + a_2)^{3/2}] \\ &+ R\Bigg(2e\rho - \frac{1}{3}\left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \\ &- \frac{1}{3}[(e^2\rho^2 + a_1)^{1/2} + (e^2\rho^2 + a_2)^{1/2}]\Bigg) \\ &+ \left(\frac{R^2}{40} + \frac{1}{60}R_{\mu\nu}R^{\mu\nu} + \frac{1}{30}\Box R\right)\Bigg(\frac{1}{(m^2 - (\lambda'/6)\rho^2 + (\eta/120)\rho^4)^{1/2}} \\ &+ \frac{1}{e^2\rho^2}[(e^2\rho^2 + a_1)^{1/2} + (e^2\rho^2 + a_2)^{1/2}]\Bigg) \\ &+ \Bigg(\frac{1}{2(m^2 - (\lambda'/6)\rho^2 + (\eta/120)\rho^4)^{3/2}} \\ &+ \frac{1}{(e^2\rho^2)^3}[(e^2\rho^2 + a_1)^{3/2} + (e^2\rho^2 + a_2)^{3/2}]\Bigg)E_{3c} \\ &+ \frac{1}{e\rho}\left(-\frac{R^2}{3} + \frac{1}{6}R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}\Box R\right) + \frac{1}{2(e\rho)^3}E'_3 + \cdots \Bigg\} \end{aligned} \tag{65}$$

$$
\begin{aligned}
a_1 = \frac{v}{2}\left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \Bigg[& \left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \\
& + \left(4e^2\rho^2 + m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \Bigg] \\
a_2 = \frac{v}{2}\left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \Bigg[& \left(m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \\
& - \left(4e^2\rho^2 + m^2 - \frac{\lambda'}{6}\rho^2 + \frac{\eta}{120}\rho^4\right)^{1/2} \Bigg].
\end{aligned}
\tag{66}
$$

It is seen that the $f^{(n)}$ terms in (50) containing derivatives of the χ fields do not contribute for constant χ. It is clear that in this case the requirement that the perturbative expansion of the effective potential be well defined restricts the range of the parameters to $\lambda' > \eta\rho^2/10$ and $m^2 > \eta\rho^4/60$. We also find that the mean-field perturbation theory technique used here does not give satisfactory and consistent results for the case when m^2 is negative, for the specific theory we have considered. It should be emphasized that the choice of the Lagrangian (5), which expresses the original ϕ^6 theory in (4) in terms of the auxiliary composite field χ, is by no means unique. There exist families of Lagrangians written in terms of χ to which the theory in (4) is equivalent.

3. Discussion

Our attempt has been to include contributions from all saddle points of the scalar field through the auxiliary mean-field-theory approach to obtain a result in curved spacetime which is reparametrization invariant, gauge invariant and gauge condition independent, both on as well as off the mass shell, by adopting the Vilkovisky–DeWitt procedure. Our method of inclusion of the saddle-point contributions by performing the ϕ integration first differs from the method followed in [14]. Since we have considered a gauged ϕ^6 theory, our result (35) also contains the vector contributions to the effective potential. The extra terms in our final result arising because of our procedure also include the Vilkovisky terms (those terms which are multiplied by the factor v) which are necessary to ensure that the same result is obtained in all gauges. We have performed a local expansion in powers of the inverse effective mass and obtained χ in terms of ϕ only to the lowest order in ϵ. Calculation of non-local terms in the effective action can be performed following the approach in [29]. It would be interesting to see the effect of the $O(\epsilon)$ terms of χ in the effective action in broken symmetry situations. In these situations, terms containing derivatives of the scalar field and of the field χ (for instance, in the $f^{(n)}$ terms) would become important, particularly if the background spacetime is non-trivial. An elegant way of calculating the gradient terms in the effective action has been described in [30]. This work takes inspiration from the work of Bunch and Parker [31], who use Riemann normal coordinates to obtain a momentum space representation of the Feynman propagator for quantum fields in curved space. We have limited ourselves only to the constant (non-derivative) terms of the effective action, as the calculations become too involved and cumbersome otherwise.

Acknowledgments

We are grateful to Dr D J Toms for going through the manuscript and for very helpful comments JB would like to thank Professor Ashok Das for informing her about [5]. Most of this work was made possible with support some time ago from CSIR, New Delhi, India, which JB would like to acknowledge.

References

[1] Heisenberg W and Euler H 1936 *Z. Phys.* **98** 714
Weisskopf V 1936 *K. Danske Vidensk. Selesk. Mat. -fys. Medd.* **14** 6
Schwinger J 1951 *Phys. Rev.* **82** 664
Goldstone J, Salam A and Weinberg S 1962 *Phys. Rev.* **127** 965
Jona-Lasinio G 1964 *Nuovo Cimento* **34** 1790

[2] Dittrich W and Reuter M 1985 *Effective Lagrangians in Quantum Electrodynamics (Lecture Notes in Physics 220)* (Berlin: Springer)

[3] Parker L 1979 *Recent Developments in Gravitation (Cargese 1978 Lectures)* ed M Levy and S Deser (New York: Plenum)

[4] Adler S L 1982 *Rev. Mod. Phys.* **54** 729

[5] David F 1984 *Phys. Lett.* **138B** 383

[6] Vilkovisky G A 1984 *Nucl. Phys.* B **234** 125

[7] DeWitt B S 1987 *Quantum Field Theory and Quantum Statistics* ed I A Batalin, C J Isham and G A Vilkovisky (Bristol: Adam Hilger)

[8] DeWitt B S 1981 *Quantum Gravity II* ed C J Isham, R Penrose and D W Sciama (Oxford: Oxford University Press)

[9] Burgess C P and Kunstatter G 1987 *Mod. Phys. Lett.* A **2** 875

[10] Symanzik K 1970 *Commun. Math. Phys.* **16** 48

[11] O'Raifeartaigh L and Parravicini G 1976 *Nucl. Phys.* B **111** 501

[12] Stevenson P M 1985 *Phys. Rev.* D **30** 1712

[13] Balakrishnan J and Moss I G 1994 *Phys. Rev.* D **49** 4113

[14] Townsend P K 1975 *Phys. Rev.* D **12** 2269
Townsend P K 1976 *Phys. Rev.* D **14** 1715
Townsend P K 1977 *Nucl. Phys.* B **118** 199

[15] Giddings S, Abbott J and Kuchar K 1984 *Gen. Rel. Grav.* **16** 751

[16] Deser S, Jackiw R and 't Hooft G 1984 *Ann. Phys.* **152** 220

[17] Bender C M, Cooper F and Guralnik G S 1977 *Ann. Phys.* **109** 165
Bender C M and Cooper F 1983 *Nucl. Phys.* B **224** 403
Cooper F, Guralnik G S and Kasdan S 1976 *Phys. Rev.* D **14** 1607

[18] Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 2804

[19] Cornwall J M, Jackiw R and Tomboulis E 1974 *Phys. Rev.* D **10** 2428

[20] Toms D J 1988 *Proc. 2nd. Canadian Conf. on General Relativity and Relativistic Astrophysics* ed A Coley, C Dyer and T Tupper (Singapore: World Scientific)

[21] DeWitt B S 1965 *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)

[22] Kunstatter G 1991 *Gravitation: A Banff Summer Institute* ed R Mann and P Wesson (Singapore: World Scientific)

[23] Toms D J 1995 Private communication

[24] Kunstatter G 1987 *Super Field Theories (Proc. NATO Study Institute, Vancouver, Canada, 1986) (NATO ASI Series B 160)* ed H C Lee *et al* (New York: Plenum)

[25] Balakrishnan J and Toms D J 1992 *Phys. Rev.* D **46** 4413

[26] Russell I H and Toms D J 1989 *Phys. Rev.* D **39** 1735

[27] Hawking S W 1977 *Commun. Math. Phys.* **55** 133

[28] Gilkey P 1975 *J. Diff. Geom.* **10** 601

[29] Parker L and Toms D J 1985 *Phys. Rev.* D **31** 953
Parker L and Toms D J 1985 *Phys. Rev.* D **31** 3424
Jack I and Parker L 1985 *Phys. Rev.* D **31** 2439

[30] Moss I G, Toms D J and Wright A 1992 *Phys. Rev.* D **46** 1671

[31] Bunch T S and Parker L 1979 *Phys. Rev.* D **20** 2499